On the amount of abelian groups of a given order and a related number theoretic problem

Introduction

The starting point of our observations will be the asymptotic behavior of the amount of different (i.e. non-isomorphic) abelian groups of order n for large n. We will denote this amount by $f(n)$, and show that

$$\sum_{k=1}^{n} f(k) = An + O(\sqrt{n}) \tag{1}$$

(the O-relations are wrt the limit as $n \to \infty$ and are not necessarily evenly distributed in the parameters) where

$$A = \zeta(2)\zeta(3)\zeta(4)\dots$$

($\zeta(s)$ is the Riemann zeta function). The value of A lies between 2 and 2.5. (1) implies that

$$\frac{1}{n}\sum_{k=1}^{n} f(k) \to A \quad \text{as} \quad n \to \infty$$

which means that, *on average there are A different abelian groups of order n*.
We will furthermore apply the in (1) used method in §2, to asymptotically estimate another number theoretic function. To clarify the method used here, we will formulate it in a generalized lemma.
Let $\psi(n)$ be a number theoretic function which we want to estimate. Our method will be determining another number theoretic function $\omega(n)$ together with a positive integer i, which is connected to $\psi(n)$ through the formula†

$$\psi(n) = \sum_{l=1}^{n} \omega(l) \left[\sqrt[i]{\frac{n}{l}}\right] \tag{2}$$

We will then correlate the asymptotic behavior of $\psi(n)$ with the summatoric function $\chi(n) = \sum_{l=1}^{n} \omega(l)$.
The most important case of going from $\chi(n)$ to $\psi(n)$ will be expressed by the following lemma.
If (2) and

$$\chi(n) = \sum_{l=1}^{n} \omega(l) = O(\sqrt[i+1]{n}) \tag{3}$$

then

$$\psi(n) = C\sqrt[i]{n} + O(\sqrt[i+1]{n})$$

where C is the constant

$$C = \sum_{l=1}^{\infty} \frac{\omega(l)}{\sqrt[i]{l}} \tag{4}$$

(In our applications $\omega(n)$ will be strictly positive, thus $C > 0$)
Proof Generally known, (3) implies that the Dirichlet series

$$\sum_{l=1}^{\infty} \frac{\omega(l)}{l^s}$$

converges for $s > \frac{1}{i+1}$ with remainder

$$\sum_{l=1}^{\infty} \frac{\omega(l)}{l^s} = O(n^{\frac{1}{i+1}-s})$$

In particular, the series (4) also converges, and we have

$$\sum_{l=1}^{n} \frac{\omega(l)}{\sqrt[i]{l}} = C + O(n^{\frac{1}{i+1}-\frac{1}{i}})$$

Thus due to (2) and (3)

$$\begin{aligned}
\psi(n) &= \sum_{l=1}^{n} \omega(l)\sqrt[i]{\frac{n}{l}} + \sum_{l=1}^{n} \omega(l) \left\{ \left[\sqrt[i]{\frac{n}{l}}\right] - \sqrt[i]{\frac{n}{l}} \right\} \\
&= \sqrt[i]{n} \sum_{l=1}^{n} \frac{\omega(l)}{\sqrt[i]{l}} + O\left(\sum_{l=1}^{n} \omega(l)\right) \\
&= \sqrt[i]{n} \left\{ C + O(n^{\frac{1}{i+1}-\frac{1}{i}}) \right\} + O(\sqrt[i+1]{n}) \\
&= C\sqrt[i]{n} + O(\sqrt[i+1]{n})
\end{aligned}$$

as stated.

§1 Amount of abelian groups of a given order

In the introduction we denoted the amount of different abelian groups of order n by $f(n)$. For the purpose of estimating the summatoric function of $f(n)$, we will express $f(n)$ in a separate form.
It is known (cf. e.g. A. Speiser, *Die Theorie der Gruppen von endlicher Ordnung*, II. edition (Berlin, 1927), page 51; or H. Hasse, *Aufgebansammlung zur höheren Algebra* (Berlin und Lepzig, Sammlung Göschen, 1934), page 95) that there are as many different abelian groups of order n as there are different ways of writing n as a produt

of prime powers, without respect to ordering.
It is expedient to consider the following generalization of the number theoretic function $f(n)$:
We will denote by $f_i(k)$ the number of different ways of writing k as a product of prime powers, without respect to ordering, if only prime powers of exponent $\geq i$ are taken into account. It is clear, that the number theoretic function $f_1(k)$ is identitcal to the previously defined $f(k)$. Furthermore, if we consider an empty product to be 1, then $f_i(1) = 1$ for every i.
In particular we would like to prove the following relation:

$$f_i(k) = \sum_{d^i|k} f_{i+1}\left(\frac{k}{d^i}\right) \tag{5}$$

We will start by showing it in the case of $k = p^\alpha$, i.e. a prime power. For this purpose we will show that

$$f_i(p^\alpha) = f_{i+1}(p^\alpha) + f_i(p^{\alpha-i}) \tag{6}$$

By definition $f_i(p^\alpha)$ means indeed the amount of solutions to the equation

$$\begin{aligned} &p^\alpha = p^{\alpha_1+\alpha_2+\dots} \\ &i \leq \alpha_1 \leq \alpha_2 \leq \dots \end{aligned} \tag{7}$$

The solutions are partly such that $i+1 \leq \alpha_1 \leq \alpha_2 \leq \dots$, but also partly such that $i = \alpha_1 \leq \alpha_2 \leq \dots$; the former amount is by definition $f_{i+1}(p^\alpha)$, and the latter being $f_i(p^{\alpha-i})$, which proves the correctness of (6).
To conclude (5) for prime powers from this, we will assume that the statement holds for $p^{\alpha-i}$, i.e.

$$f_i(p^{\alpha-i}) = f_{i+1}(p^{\alpha-i}) + f_{i+1}(p^{\alpha-2i}) + \dots \tag{8}$$

Then because (6)

$$f_i(p^\alpha) = f_{i+1}(p^\alpha) + f_i(p^{\alpha-i}) = f_{i+1}(p^\alpha) + f_{i+1}(p^\alpha - i) + \dots$$

therefore the formula (5) also holds for p^α. Thus by induction (5) holds for prime powers.
To fully prove it, we still have to show that when (5) holds for k and l (where $\gcd(k,l) = 1$), then it will also hold for kl.
From the definition we immediately get $f_i(kl) = f_i(k)f_i(l)$ for $(k,l) = 1$. If we further assume that (5) holds for k and l, then

$$\begin{aligned} f_i(kl) = f_i(k) f_i(l) &= \sum_{d^i|k} f_{i+1}\left(\frac{k}{d^i}\right) \sum_{e^i|k} f_{i+1}\left(\frac{l}{e^i}\right) \\ &= \sum_{d^i|k, e^i|l} f_{i+1}\left(\frac{k}{d^i}\right) f_{i+1}\left(\frac{l}{e^i}\right) \\ &= \sum_{d^i|k, e^i|l} f_{i+1}\left(\frac{kl}{d^i e^i}\right) = \sum_{g^i|kl} f_{i+1}\left(\frac{kl}{g^i}\right) \end{aligned}$$

with which we have now proven (5) in general.

We now sum both sides of (5) for $k = 1, 2, \ldots, n$ and get

$$\sum_{k=1}^{n} f_i(k) = \sum_{k=1}^{n} \sum_{d^i|k} f_{i+1}\left(\frac{k}{d^i}\right) = \sum_{l=1}^{n} f_{i+1}(l) \left[\sqrt[i]{\frac{n}{l}}\right] \tag{9}$$

To apply our lemma, we must first show

$$\sum_{l=1}^{n} f_{i+1}(l) = O(\sqrt[i+1]{n})$$

For this purpose we will need the following lemma.

The series

$$\sum_{l=1}^{\infty} \frac{f_{i+1}(l)}{\sqrt[i]{l}}$$

converges; its sum is

$$\sum_{l=1}^{\infty} \frac{f_{i+1}(l)}{\sqrt[i]{l}} = \zeta\left(1 + \frac{1}{i}\right) \zeta\left(1 + \frac{2}{i}\right) \ldots \tag{10}$$

where $\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots$ is the Riemann ζ-function for $s > 1$.

Proof We will first show that the infinite product A_i on the right side of (10) is convergent. For $i = 1$ we have

$$A_1 = \zeta(2)\zeta(3)\ldots = (1 + (\zeta(2) - 1))(1 + (\zeta(3) - 1))\ldots$$

and the series

$$\sum_{k=2}^{\infty} (\zeta(k) - 1) = \sum_{k=2}^{\infty} \sum_{n=2}^{\infty} \frac{1}{n^k} = \sum_{k=2}^{\infty} \frac{1}{n(n-1)}$$

converges, namely to 1. (This also implies that $A_1 > 2$ by the way) However, because

$$\zeta\left(2 + \frac{k}{i}\right) \leq \zeta\left(2 + \left[\frac{k}{i}\right]\right)$$

the product A_i will, besides the first $i-1$ factors, be majorized through the ith power of the product A_1, therefore A_i is also convergent.
It is known that for $s > 1$, $\zeta(s)$ has the Euler product representation

$$\zeta(s) = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right) \tag{11}$$

which is to be taken over all primes.
We thus get that

$$\begin{aligned} A_i &= \zeta\left(\frac{i+1}{i}\right)\zeta\left(\frac{i+2}{i}\right)\dots \\ &= \prod_p \left(1 + \frac{1}{p^{\frac{i+1}{i}}} + \frac{1}{p^{\frac{2(i+1)}{i}}} + \dots\right) \prod_p \left(1 + \frac{1}{p^{\frac{i+2}{i}}} + \frac{1}{p^{\frac{2(i+2)}{i}}} + \dots\right)\dots \end{aligned}$$

All factors of the product are > 1 and the product is, like we have shown already, convergent. Just like the infinite series with positive summands in each factor; we can therefore do the multiplication factor-wise and build partial products.
Then $\frac{1}{n^{\frac{1}{i}}}$ appears as many times as n can be represented in the form

$$p_1^{\alpha_{11}(i+1)+\alpha_{12}(i+2)+\dots} p_2^{\alpha_{21}(i+1)+\alpha_{23}(i+2)+\dots} \dots = p_1^{\overbrace{(i+1+(i+1)+\dots}^{\alpha_{11}}+\overbrace{(i+2+(i+2)+\dots}^{\alpha_{12}}+\dots)} p_2^{\overbrace{(i+1)+(i+1)+\dots}^{\alpha_{21}}+\dots} \dots$$

i.e. as many times as we can break n down into a product of prime powers with exponent $\geq i+1$, i.e. $f_{i+1}(n)$ many. We have thus shown (10).
We will now show that

$$s_i(n) = \sum_{k=1}^{n} f_i(k) = O(\sqrt[i]{n}) \tag{12}$$

Due to (9), (10), and $f_i(l) \geq 0$ we have

$$\sum_{k=1}^{n} f_i(k) \leq \sum_{l=1}^{n} f_{i+1}(l)\sqrt[i]{\frac{n}{l}} = \sqrt[i]{n}\sum_{l=1}^{n} \frac{f_{i+1}(l)}{\sqrt[i]{l}} \leq A_i\sqrt[i]{n}$$

which immediately implies (12). (12) also allows us to conclude that

$$\sum_{k=1}^{n} f_{i+1}(k) = O(\sqrt[i+1]{n})$$

which then in turn allows us to use our lemma from the introduction. We obtain

$$s_i(n) = \sum_{k=1}^{n} f_i(k) = A_i\sqrt[i]{n} + O(\sqrt[i+1]{n}) \tag{13}$$

For the special case $i = 1$, our result is

$$\sum_{k=1}^{n} f(k) = A_1 n + O(\sqrt{n})$$

just like what was talked about in the introduction.

§2 Distribution of numbers k with $f_i(k) \neq 0$

It is immediately clear that $f_i(k)$, the amount of decompositions of k as a product of prime powers whose exponent is $\geq i$, will only be non-null for numbers k, in which all primes have exponent $\geq i$. For the sake of brevity, we will call these numbers "ith type integers".
We now want to asymptotically evaluate the amount of ith type integers up to n. We will denote this amount by $\psi_i(n)$. Apparently $\psi_i(n) \leq s_i(n)$, as $s_i(n) = \sum_{k=1}^{n} f_i(n)$ is a sum of non-negative integers, who has $\psi_i(n)$ summands with ≥ 1. Thereby from (12) we can conclude that

$$\psi_i(n) = O(\sqrt[i]{n}) \tag{14}$$

We will more specifically show that *the asymptotic formula*

$$\psi_i(n) = C_i \sqrt[i]{n} + O(\sqrt[i+1]{n})$$

holds, where C_i is a positive integer dependant on i but not n.
For this purpose we will denote the numbers whose prime factorization only contains primes with an exponent $\geq i$ but $\leq 2i$ by

$$a_1, a_2, a_3, \ldots \tag{15}$$

(e.g. in increasing order). If $\chi_i(n)$ is the amount of elements $\leq n$ in the sequence (15), then $\chi_i(n) \leq \psi_{i+1}(n)$, as the sequence (15) contains many $i+1$th type integers. Therefore by (14) we have

$$\chi_i(n) = O(\sqrt[i+1]{n}) \tag{16}$$

Now every ith type integer can be expressed as one and only one product of two of such factors, where the former is an ith power, and where the latter is contained in the sequence (15). As a matter of fact, if

$$k = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r} \quad (\alpha_1, \alpha_2, \ldots, \alpha_r, \geq i)$$

is an arbitrary ith type integer, then every exponent α_j can be uniquely expressed in the form of $\alpha_j = \beta_j i + \gamma_j$ with $\beta_j \geq 0, \gamma_j = 0$ or $i < \gamma_j < 2i$, and then

$$k = (p_1^{\beta_1} p_2^{\beta_2} \ldots p_r^{\beta_r})^i \cdot p_1^{\gamma_1} p_2^{\gamma_2} \ldots p_r^{\gamma_r}$$

is the only decomposition of the wanted form. Naturally the product of an ith power together with a number in (15) will be an ith type integer.
Thus

$$\psi_i(n) = \sum_{m^i a_j \leq n} 1 = \sum_{a_j \leq n} \sum 1 = \sum_{a_j \leq n} \left[\sqrt[i]{\frac{n}{a_j}} \right] \tag{17}$$

where $m \leq \sqrt[i]{\frac{n}{a_j}}$ and a_j are the elements in the sequence (15).
By (16) and (17) we can apply our lemma from the introduction to $\psi_i(n)$; we set $\omega(n) = 1$ or 0, depending on whether n is contained in (15). We obtain

$$\psi_i(n) = C_i \sqrt[i]{n} + O(\sqrt[i+1]{n})$$

where

$$C_i = \sum_{j=1}^{\infty} \frac{1}{\sqrt[i]{a_j}} > 0$$

We have therefore shown the claim.

Footnotes

† The easily proven fact, that to every $\psi(n)$ and i there exists only one such $\omega(w)$, given by

$$\omega(n) = \sum_{d^i | n} \mu(d) \left\{ \psi\left(\frac{n}{d^i}\right) - \psi\left(\frac{n}{d^i} - 1\right) \right\}$$

where d goes through every positive integer whose ith power appears in n, is irrelevant for what we are trying to do, as we will be able to immediately determine $\omega(n)$ in every application.